Beyond the $n=0$ limit restriction for random-surface theories as derived from Hamiltonian gauge models

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Studies of the universality hypothesis for the classification of the critical behavior of statistical systems, based on the construction of gauge models as equivalent to random-surface theories, have been until now strongly restricted to the $n = 0$ (zero-spin components) limit. This difficulty is overcome here, thus rendering possible a precise description of the correspondence between the generating functions of ensembles of random surfaces and the (more simple) partition functions of the related Hamiltonian gauge models, for a wide spectrum of physically relevant processes.

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One of the most basic clues for the classification of the critical behavior of statistical systems is provided by the universality hypothesis, which states that the values of the critical exponents characterizing a system are uniquely determined by a few fundamental attributes, such as the spatial dimensionality of the system, its symmetry, etc. In a very recent paper, Banavar, Maritan, and Stella have been able to prove that topology is also one of these basic attributes [1].

A very important role in the study of those systems (appearing in such diverse areas as material sciences, statistical physics, biology, and high-energy physics) is played by random surfaces in their different versions, the main ones being triangulated random surfaces and random surfaces on the lattice. We shall here deal with the latter ones only and, the lattice will be a cubic one. The surfaces are built from elementary plaquettes and, when they are closed and self-avoiding, each bond (edge) belongs to exactly two plaquettes. The area of the surface is an integer number, in terms of the elementary plaquette area, and the volume enclosed by the surface is also an integer number of elementary cube volumes. Without loss of generality, we shall take the lattice constant equal to 1.

Triangulated random surfaces with the topology of a sphere were rigorously proven to be trivial (i.e., noninteracting) by Durhuus, Fröhlich, and Jónsson some years ago [2]. An additional problem was the appearance of spikes (infinities), for any kind of triangulation [3]. This latter difficulty was the first to be solved, by considering self-avoiding surfaces, by modifying the surface action [4], and by dealing with multicoverings [5]. Concerning triviality, it has been interpreted by saying that some entropic mechanism favors branched polymerlike configurations, so that one always falls into this universality class when the topology of the surface is that of a sphere (zero number of handles) or even finitely more complicated (i.e., with a bounded number of handles), although this last result is still lacking a rigorous proof. It has been demonstrated in [1] that, on the contrary, one falls into a different universality class when one considers random surfaces with an unbounded number of handles.

The important results in [1] rely very heavily upon previous work of Maritan and Stella [6], in which it was shown how the generating function corresponding to the grand canonical ensemble of lattice random surfaces can be derived from a gauge model defined by a reduced Hamiltonian, in the limit when the number n of components of the vectors attached to the bonds goes to zero [see Eq. (4) below]. Such a procedure simplifies the calculations enormously; even more, it actually gives sense to the sum over surfaces, allowing one to study the random-surface problem by analyzing the corresponding gauge model. It turns out, however, that this approach has been possible *only* in the limit $n \to 0$ and, moreover, only after making a mathematically arguable, *ad hoc* step. Such longstanding problem will be resolved here in a quite natural way, which will render possible a precise description of the correspondence between the generating functions of grand canonical ensembles of random surfaces and the partition functions of the related Hamiltonian gauge models.

This connection between random surfaces and gauge models is reminiscent of the very important one which exists between the s-state Potts model and the random resistor networks in the limit $s \to 0$. Also, in order to make the connection precise, it is necessary to scale the coupling in a very particular manner. The interest of establishing rigorous and precise links between gauge models and random surfaces is in no way less. It can also be viewed as a way of generalizing the de Gennes theorem [7], which connects self-avoiding random walks and n-component spin models in the limit $n \to 0$ and has played such a fundamental role in the development of field-theoretical descriptions of polymers.

Being more concrete, the generating function corresponding to random surfaces of a given topology can be written as

$$G(K) = \sum_{S}{}' K^{|S|}, \tag{1}$$

where the prime means that the sum is extended over

the set of surfaces of vesicles on the lattice with the constraint that a given plaquette belongs to the reference surface, and $|S|$ is the surface area. Notice that, in order to avoid complications which are irrelevant for our argument, we limit ourselves to the zero-pressure case of [1]. Denoting by C, P, and b the elementary cubes, plaquettes, and bonds of the lattice, and by ∂C, ∂P, and ∂b their corresponding boundaries, respectively, let us now consider the gauge model defined by the reduced Hamiltonian [1,6]

$$H = K\sum_P \sigma_P \sum_{\alpha=1}^{n} \prod_{b\in\partial P} s_b^\alpha + u\sum_C \prod_{P\in\partial C} \sigma_P, \tag{2}$$

where the $\mathbf{s}_b = (s_b^\alpha)$ are n-component vectors of modulus $\sqrt{n}$ attached to the bonds and the $\sigma_P = \pm 1$ are Ising variables attached to the plaquettes of the lattice. This Hamiltonian is invariant under the following gauge transformation:

$$s_b^\alpha \longrightarrow \epsilon_b\, s_b^\alpha, \quad \sigma_P \longrightarrow \left(\prod_{b\in\partial P}\epsilon_b\right)\sigma_P, \quad \epsilon_b = \pm 1. \tag{3}$$

Expanding the partition function $Z(K)$ associated with the Hamiltonian gauge model above (for $u \to \infty$), and calculating the traces involved, one can check the following relation between $Z(K)$ and the generating function $G(K)$ of Eq. (1):

$$\lim_{n\to 0}\left[\frac{K}{nN}\frac{\partial}{\partial K}\ln Z(K)\right] = G(K) + 3K^2, \tag{4}$$

N being the total number of sites of the lattice. By analyzing the critical exponents corresponding to the gauge model, and by using this equivalence, it has been demonstrated in [1] that the universality class associated with lattice random surfaces of unbounded genus is not the same as the one corresponding to spherical surfaces. However, as pointed out by the authors themselves, their analysis could be carried out only in the $n \to 0$ limit, and not first treating the arbitrary n case and then letting n tend to 0.

In other words, using the formulation developed in [6], the preceding equivalence (4) between the Hamiltonian model and the one of closed, self-avoiding random surfaces with an unrestricted number of handles is only obtained in the (hardly intuitive) limit $n \to 0$. The basic equations leading to this limit are

$$\begin{aligned}
&\sum_{\alpha=1}^{n}(s_b^\alpha)^2 = n, \quad s_b^\alpha s_b^\beta = \delta^{\alpha\beta}(s_b^\alpha)^2,\\
&(s_b^\alpha)^k = n\,(s_b^\alpha)^{k-2}, \quad k \geq 3,\\
&\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^2 \equiv \frac{1}{2n}\sum_{s_b}(s_b^\alpha)^2 = 1,\\
&\lim_{n\to 0}\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^{2k} = \delta_{k0} + \delta_{k1}.
\end{aligned} \tag{5}$$

The origin of these various conditions is the following. As a whole, their physical meaning is similar to that of the corresponding ones which —in the case of the de Gennes theorem [7]— give rise to the appearance of the n-component spin vector in the self-avoiding random-walk models. Namely, s_b^α is an n-component vector attached to the bond b of the hypercubic lattice, which can take $2n$ orientations (i.e., plus and minus each of the basis axes). However, being more specific, these equations (which were introduced in [6]) have diverse nature.

The first, third, and fourth equations are quite natural: they just amount to a convenient normalization (to $\sqrt{n}$) of the spin vector (aside from taking into account its $2n$ possible orientations). But the second equation is a rather strong restriction imposed on the possible form of the n vectors $\mathbf{s}_b$. In particular, a vector such as $\mathbf{s}_b = (1,1,\ldots,1)$ is not allowed in this formulation, and only those of the kind

$$\mathbf{s}_b = (0,0,\ldots,\pm\sqrt{n},\ldots,0) \tag{6}$$

remain. Notice that there are $2n$ of such vectors, $\mathbf{s}_{b,\pm\alpha}$, $\alpha = 1,2,\ldots,n$, and that their components are given by

$$\mathbf{s}_{b,\pm\alpha}^\beta = \pm\sqrt{n}\,\delta^{\alpha\beta}. \tag{7}$$

This notation is more precise than the traditional one [employed in (5)], but we shall mostly use this last one (unless strictly necessary) in order not to confuse the reader who is already familiar with Refs. [1] and [6].

The last of Eqs. (5) deserves a special comment. While the fact that $\lim_{n\to 0}\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^2 = 1$ is an immediate consequence of the first and third, the rest of the statement, i.e., the fact that

$$\lim_{n\to 0}\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^{2k}\Big|_{k=0} = 1, \tag{8}$$

is by no means a direct consequence of the first four equations (5). Actually, what one gets from them is

$$\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^{2k} = n^{k-1}, \tag{9}$$

which is clearly divergent for $k < 1$. This difficulty cannot be resolved, in principle, whatever the meaning attributed to the delta function δ_{k0} in the last of Eqs. (5), which must in fact be considered as a convenient *guess* leading to the desired results.

Since we approach the value $n = 0$ from integer values of n (the number of components of the vectors attached to the bonds of the lattice), it seems clear that in any mathematically well-founded theory yielding the limit $n \to 0$, one will necessarily need to analytically continue the variable n and, moreover, the result of the limit will be given in terms of Dirac δ functions. This is exactly what happens, as we are going to see. We look for a theory which should make perfect sense for finite values of n and which must reduce to the "guessed" limit [last equation of (5)] when $n \to 0$ (with the Kronecker δ functions replaced by Dirac ones). There is a natural way to solve this problem, by choosing a convenient representation of the Dirac δ distribution which will do the job. What is not immediate at all (to our knowledge nobody has succeded in this yet) is managing to preserve the full theory when n is maintained different from 0 (whatever small it be). This will be here resolved by changing the

dependence of the components of the vectors $\mathbf{s}_b$ on n, and also by letting the number of components of these vectors be a convenient function of n (not trivially n).

Let us be precise. The main idea involved in the work is to find a couple of smooth functions f and g such that

$$s_b^\alpha = (0, 0, \ldots, \pm f(n), \ldots, 0)_{g(n)} \tag{10}$$

satisfies the last of Eqs. (5). Of course, (10) is a *formal* expression [to be directly compared with (6)] which conveys the intuition behind the (actually rigorous) procedure. That means we shall work with spin vectors of the kind (6) (i.e., with only one nonzero component), defined on a space of vectors of $g(n)$ components. This will be obtained by making the Cartesian product of R (the one-dimensional real line) a number $g(n)$ of times—which has an absolutely precise mathematical meaning [8]. Intuitively, and for the sake of comparison, we can think of this vector as having $[g(n) + \epsilon]$ components ([ ] means here integer part, and ϵ is arbitrarily small), only one of which is nonzero. This is how expression (10) has to be viewed (formally), and constitutes the convenient implementation of the concept of "limit $n \to 0$" of expression (6), which was freely introduced in [6,1] without any mathematical basis. The virtue of our correct mathematical definitions is that they will allow us to take this limit $n \to 0$ in a rigorous way.

Summing up, the correct expression —the one which substitutes (7)— is [8]

$$s_{b,\pm\alpha}^\beta = \pm f(n)\, \delta(\alpha - \beta), \quad \alpha, \beta \in [0, g(n)]. \tag{11}$$

Introducing a convenient representation for the δ functions, after several calculations we obtain the result [9]

$$f(n) = \exp\left(-\frac{1}{n}\right), \qquad g(n) = n\left[1 + \exp\left(\frac{2}{n}\right)\right]^{-1}. \tag{12}$$

In this expression we can already see the reason why the model of Refs. [1,6] was not well suited to extend the $n = 0$ results to $n \neq 0$. In fact, the behavior of the functions which give the nonzero component and the number of dimensions of the spin-vector space turns out to be exponential (in $1/n$), rather than linear or powerlike (as was implicitly assumed in that model). The physical reason for this behavior is not clear (but it is not surprising either [10]). Actually, when computing the different powers $(s_b^\alpha)^{2k}$ this form of the functions is valid for $k \geq 1$, and must be supplemented with the following additional expressions for values of k between 0 and 1:

$$f(n) \simeq \exp\left(-\frac{1}{n}\right), \quad g(n) \simeq n, \quad 0 < k < \frac{1}{2},$$
$$f(n) \simeq \exp\left(\frac{1}{n}\right), \quad g(n) \simeq n \exp\left(\frac{2}{n}\right), \quad \frac{1}{2} < k < 1. \tag{13}$$

Compared with the previous theory [1,6], in the new model the value of the components of $\mathbf{s}_b$, and also their number, decay much more quickly as $n \to 0$ (exponentially versus powerlike behavior). This is one of the clues in the derivation of our finite-n model. There is also an additional difference which corresponds to the range $1/2 < k < 1$. In this case, both the number and the value of the nonzero component of the bond vectors tend to infinity, as $n \to 0$. However, here also the trace per component of the $2k$ power of the vector falls exponentially to zero (and with exactly the same speed), so that the last of Eqs. (5) is satisfied. Notice that the trace per component, which in [1,6] was given as

$$\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^{2k} = \frac{1}{2n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}(s_{b,\beta}^\alpha)^{2k} \tag{14}$$

[this is the one which appeared in Eqs. (5)], is generalized here [in natural accordance with the preceding expressions (10)] by means of the formula

$$\mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^{2k} = \frac{1}{2g(n)}\int_0^{g(n)} d\alpha\, \delta(\alpha - \beta)\, f(n)^{2k}. \tag{15}$$

Equations (11)–(13) provide both the analytical continuation of the integer powers of the bond vectors s_b^α and also the continuation of the important $n = 0$ results to finite values of n. It is interesting to observe that one can even get the *Kronecker δ functions* as they stand in the last of Eqs. (5).

Our procedure is mathematically rigorous and can be extended easily in order to include more elaborate transformations, of the kind

$$\lim_{n\to 0} \mathrm{Tr}_{\{s_b\}}(s_b^\alpha)^{2k} = \delta_{k0} + \delta_{k1} + \delta_{k2}, \tag{16}$$

and so on. The functions f and g above here have the following form:

$$f(n) \simeq \exp\left(-\frac{1}{n}\right), \quad g(n) \simeq n \exp\left(-\frac{a}{n}\right), \quad a < k < a + \frac{1}{2}, \quad a = 0, 1,$$
$$f(n) \simeq \exp\left(\frac{1}{n}\right), \quad g(n) \simeq n \exp\left(\frac{b}{n}\right), \quad b - \frac{1}{2} < k < b, \quad b = 1, 2,$$
$$f(n) = \exp\left(-\frac{1}{n}\right), \quad g(n) = n\left[1 + \exp\left(\frac{2}{n}\right) + \exp\left(\frac{4}{n}\right)\right]^{-1}, \quad k > 2. \tag{17}$$

The preceding technique makes possible the analysis of the correspondence when dealing with more involved Hamiltonian gauge models in a mathematically rigorous way. Examples of such models (with their respective local gauge invariances) are the following:

$$H_1 = K \sum_P \sum_\alpha \sum_\beta \sigma_P^\beta \prod_{b\in\partial P} s_b^\alpha + u \sum_C \sum_\beta \prod_{P\in\partial C} \sigma_P^\beta,$$

$$\mathbf{s}_b \longrightarrow \epsilon_b \mathbf{s}_b, \quad \sigma_P \longrightarrow \left(\prod_{b\in\partial P} \epsilon_b\right) \sigma_P. \quad (18)$$

this corresponds essentially to several copies of the original model [1], but here one has the possibility of also playing with the trace over the spin vectors σ_P associated with the plaquettes, and

$$H_2 = K \sum_P \sum_\alpha \sigma_P^\alpha \prod_{b\in\partial P} s_b^\alpha + u \sum_C \sum_\alpha \prod_{P\in\partial C} \sigma_P^\alpha,$$

$$s_b^\alpha \longrightarrow \epsilon_b^\alpha s_b^\alpha, \quad \sigma_P^\alpha \longrightarrow \left(\prod_{b\in\partial P} \epsilon_b^\alpha\right) \sigma_P^\alpha, \quad (19)$$

where a coupling between the bond and the plaquette vectors has been introduced. Generalization of these models to higher dimensions is also possible.

These models (and variations of them) can be employed to describe nonclosed and only-partly self-avoiding surfaces. This is done by using the generalization of the trace, as explained above [see (16)]. For instance, with

$$\lim_{n\to 0} \mathrm{Tr}_{\{s_b\}} (s_b^\alpha)^{2k} = \delta_{k0} + \delta_{k1} + \delta_{k2},$$
$$\lim_{n\to 0} \mathrm{Tr}_{\{\sigma_P\}} (\sigma_P^\alpha)^{2k} = \delta_{k0} + \delta_{k(1/2)} + \delta_{k1}, \quad (20)$$

the model (19) is capable of describing processes of bubble formation and bubble fussion. Such processes are very relevant for the description of real physical phenomena, and the scheme developed here, combined with the techniques of [1,6], allows for its study on the lattice, in terms of the corresponding Hamiltonian gauge models.

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[8] May this be the place to recall such basic concepts as the Cartesian products: $\Pi_N A = \{f : N \to A\} = \{(a_n)_{n\in N}\}$ and $\Pi_R A = \{f : R \to A\}$ (these are the sets of all successions in the arbitrary set A, and the set of all A-valued functions of a real variable, respectively). In particular, we have $\Pi_{[0,g(n)]} R = \{f : [0, g(n)] \to R\}$, $[0, g(n)]$ being a closed interval of real numbers. This last set, enlarged in order to also include distributions, is the one in which we are working.
[9] Details of the explicit derivation of expressions (12) and (13) will be given elsewhere.
[10] From a more technical point of view, the emergence of the exponential behavior can be traced back to the absolute necessity for a δ function distribution, which must play the decisive role of picking up precisely one component (the one which will be different from zero) out of each vector of the Cartesian product introduced above.